

FINITE BASIS FOR RADICAL WELL-MIXED DIFFERENCE IDEALS GENERATED BY BINOMIALS

JIE WANG

ABSTRACT. In this paper, we prove a finite basis theorem for radical well-mixed difference ideals generated by binomials. As a consequence, every strictly ascending chain of radical well-mixed difference ideals generated by binomials in a difference polynomial ring is finite, which solves an open problem of difference algebra raised by E. Hrushovski in the binomial case.

1. INTRODUCTION

In [4], E. Hrushovski developed the theory of difference scheme, which is one of the major recent advances in difference algebra geometry. In Hrushovski's theory, well-mixed difference ideals played an important role. So it is significant to make clear of the properties of well-mixed difference ideals.

It is well known that Hilbert's basis theorem does not hold for difference ideals in a difference polynomial ring. Instead, we have Ritt-Raudenbush basis theorem which asserts that every perfect difference ideal in a difference polynomial ring has a finite basis. It is naturally to ask if the finitely generated property holds for more difference ideals. Let K be a difference field and R a finitely difference generated difference algebra over K . In [4, Section 4.6], Hrushovski raised the problem whether a radical well-mixed difference ideal in R is finitely generated. The problem is also equivalent to whether the ascending chain condition holds for radical well-mixed difference ideals in R . For the sake of convenience, let us state it as a conjecture:

Conjecture 1.1. *Every strictly ascending chain of radical well-mixed difference ideals in R is finite.*

Also in [4, Section 4.6], Hrushovski proved that the answer is yes under some additional assumptions on R . In [5], A. Levin showed that the ascending chain condition does not hold if we drop the radical condition. The counter example given by Levin is a well-mixed difference ideal generated by binomials. In [9, Section 9], M. Wibmer showed that if R can be equipped with the structure of a difference Hopf algebra over K , then Conjecture 1.1 is valid. In [7], J. Wang proved that Conjecture 1.1 is valid for radical well-mixed difference ideals generated by monomials.

Difference ideals generated by binomials were first studied by X. S. Gao, Z. Huang, C. M. Yuan in [2]. Some basic properties of difference ideals generated by binomials were proved in

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that paper due to the correspondence between $\mathbb{Z}[x]$ -lattices and normal binomial difference ideals.

The main result of this paper is that every radical well-mixed difference ideal generated by binomials in a difference polynomial ring is finitely generated. As a consequence, Conjecture 1.1 is valid for radical well-mixed difference ideals generated by binomials in a difference polynomial ring.

2. PRELIMINARIES

2.1. Preliminaries for Difference Algebra. We recall some basic notions from difference algebra. Standard references are [5, 8]. All rings in this paper will be assumed to be commutative and unital.

A *difference ring*, or σ -ring for short, is a ring R together with a ring endomorphism $\sigma: R \rightarrow R$, and we call σ a *difference operator* on R . If R is a field, then we call it a *difference field*, or σ -field for short. In this paper, all σ -fields will be assumed to be of characteristic 0.

Following [3], we introduce the following notation of symbolic exponents. Let x be an algebraic indeterminate and $p = \sum_{i=0}^s c_i x^i \in \mathbb{N}[x]$. For a in a σ -ring, we denote $a^p = \prod_{i=0}^s (\sigma^i(a))^{c_i}$ with $\sigma^0(a) = a$ and $a^0 = 1$. It is easy to check that for $p, q \in \mathbb{N}[x]$, $a^{p+q} = a^p a^q$, $a^{pq} = (a^p)^q$.

Let R be a σ -ring. A σ -ideal I in R is an algebraic ideal which is closed under σ , i.e., $\sigma(I) \subseteq I$. If I also has the property that $a^x \in I$ implies $a \in I$, it is called a *reflexive σ -ideal*. A σ -prime ideal is a reflexive σ -ideal which is prime as an algebraic ideal. A σ -ideal I is said to be *well-mixed* if for $a, b \in K\{\mathbb{Y}\}$, $ab \in I$ implies $ab^x \in I$. A σ -ideal I is said to be *perfect* if for $g \in \mathbb{N}[x] \setminus \{0\}$ and $a \in K\{\mathbb{Y}\}$, $a^g \in I$ implies $a \in I$. It is easy to prove that every perfect σ -ideal is well-mixed and every σ -prime ideal is perfect.

If $F \subseteq R$ is a subset of R , denote the minimal ideal containing F by (F) , the minimal σ -ideal containing F by $[F]$ and denote the minimal well-mixed σ -ideal, the minimal radical well-mixed σ -ideal, the minimal perfect σ -ideal containing F by $\langle F \rangle$, $\langle F \rangle_r$, $\{F\}$ respectively, which are called the *well-mixed closure*, the *radical well-mixed closure*, the *perfect closure* of F respectively.

Let K be a σ -field and $\mathbb{Y} = (y_1, \dots, y_n)$ a tuple of σ -indeterminates over K . Then the σ -polynomial ring over K in \mathbb{Y} is the polynomial ring in the variables $y_i^{x^j}$ for $j \in \mathbb{N}$ and $i = 1, \dots, n$. It is denoted by $K\{\mathbb{Y}\} = K\{y_1, \dots, y_n\}$ and has a natural K - σ -algebra structure.

2.2. Preliminaries for Binomial Difference Ideals. A $\mathbb{Z}[x]$ -lattice is a $\mathbb{Z}[x]$ -submodule of $\mathbb{Z}[x]^n$ for some n . Since $\mathbb{Z}[x]$ is Noetherian as a $\mathbb{Z}[x]$ -module, we see that any $\mathbb{Z}[x]$ -lattice is finitely generated as a $\mathbb{Z}[x]$ -module.

Let K be a σ -field and $\mathbb{Y} = (y_1, \dots, y_n)$ a tuple of σ -indeterminates over K . For $\mathbf{f} = (f_1, \dots, f_n) \in \mathbb{N}[x]^n$, we define $\mathbb{Y}^{\mathbf{f}} = \prod_{i=1}^n y_i^{f_i}$. $\mathbb{Y}^{\mathbf{f}}$ is called a *monomial* in \mathbb{Y} and \mathbf{f} is called its *support*. For $a, b \in K^*$ and $\mathbf{f}, \mathbf{g} \in \mathbb{N}[x]^n$, $a\mathbb{Y}^{\mathbf{f}} + b\mathbb{Y}^{\mathbf{g}}$ is called a *binomial*. If $a = 1, b = -1$, then $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}}$ is called a *pure binomial*. A *(pure) binomial σ -ideal* is a σ -ideal generated by (pure) binomials.

For $f \in \mathbb{Z}[x]$, we write $f = f_+ - f_-$, where $f_+, f_- \in \mathbb{N}[x]$ are the positive part and the negative part of f respectively. For $\mathbf{f} \in \mathbb{Z}[x]^n$, $\mathbf{f}_+ = (f_{1+}, \dots, f_{n+})$, $\mathbf{f}_- = (f_{1-}, \dots, f_{n-})$.

Definition 2.1. A partial character ρ on a $\mathbb{Z}[x]$ -lattice L is a group homomorphism from L to the multiplicative group K^* satisfying $\rho(x\mathbf{f}) = (\rho(\mathbf{f}))^x$ for all $\mathbf{f} \in L$.

A trivial partial character on L is defined by setting $\rho(\mathbf{f}) = 1$ for all $\mathbf{f} \in L$.

Given a partial character ρ on a $\mathbb{Z}[x]$ -lattice L , we define the following binomial σ -ideal in $K\{\mathbb{Y}\}$,

$$\mathcal{I}_L(\rho) := [\mathbb{Y}^{\mathbf{f}_+} - \rho(\mathbf{f})\mathbb{Y}^{\mathbf{f}_-} \mid \mathbf{f} \in L].$$

L is called the *support lattice* of $\mathcal{I}_L(\rho)$. In particular, if ρ is a trivial partial character on L , then the binomial σ -ideal defined by ρ is called a *lattice σ -ideal*, which is denoted by \mathcal{I}_L .

Let \mathfrak{m} be the multiplicatively closed set generated by $y_i^{x_j}$ for $i = 1, \dots, n, j \in \mathbb{N}$. A σ -ideal I is said to be *normal* if for any $M \in \mathfrak{m}$ and $p \in K\{\mathbb{Y}\}$, $Mp \in I$ implies $p \in I$. For any σ -ideal I ,

$$I : \mathfrak{m} = \{p \in K\{\mathbb{Y}\} \mid \exists M \in \mathfrak{m} \text{ s.t. } Mp \in I\}$$

is a normal σ -ideal.

Lemma 2.2. A normal binomial σ -ideal is radical.

Proof. For the proof, please refer to [2]. □

In [2], it was proved that there is a one-to-one correspondence between normal binomial σ -ideals and partial characters ρ on some $\mathbb{Z}[x]$ -lattice L .

In [2], the concept of *M-saturation* of a $\mathbb{Z}[x]$ -lattice was introduced.

Definition 2.3. Assume K is algebraically closed. If a $\mathbb{Z}[x]$ -lattice L satisfies

$$(1) \quad m\mathbf{f} \in L \Rightarrow (x - o_m)\mathbf{f} \in L,$$

where $m \in \mathbb{N}$, $\mathbf{f} \in \mathbb{Z}[x]^n$, and o_m is the m -th transforming degree of the unity of K , then it is said to be *M-saturated*. For any $\mathbb{Z}[x]$ -lattice L , the smallest *M-saturated* $\mathbb{Z}[x]$ -lattice containing L is called the *M-saturation* of L and is denoted by $\text{sat}_M(L)$.

The following two lemmas were proved in [2] for the Laurent case and it is easy to generalize to the normal case.

Lemma 2.4. Assume K is algebraically closed and inversive. Let ρ be a partial character on a $\mathbb{Z}[x]$ -lattice L . If $\mathcal{I}_L(\rho)$ is well-mixed, then L is *M-saturated*. Conversely, if L is *M-saturated*, then either $\langle \mathcal{I}_L(\rho) \rangle : \mathfrak{m} = [1]$ or $\mathcal{I}_L(\rho)$ is well-mixed.

Lemma 2.5. Let K be an algebraically closed and inversive σ -field and ρ a partial character on a $\mathbb{Z}[x]$ -lattice L . Then $\langle \mathcal{I}_L(\rho) \rangle_r : \mathfrak{m}$ is either $[1]$ or a normal binomial σ -ideal whose support lattice is $\text{sat}_M(L)$. In particular, $\langle \mathcal{I}_L \rangle_r : \mathfrak{m}$ is either $[1]$ or $\mathcal{I}_{\text{sat}_M(L)}$.

3. RADICAL WELL-MIXED DIFFERENCE IDEAL GENERATED BY BINOMIALS IS FINITELY GENERATED

In this section, we will prove every radical well-mixed σ -ideal generated by binomials in a σ -polynomial ring is finitely generated as a radical well-mixed σ -ideal. For simplicity, we

only consider the case for pure binomials since it is easy to generalize the results to any binomials.

For convenience, for $h \in \mathbb{Z}[x]$, if $\deg(h_+) > \deg(h_-)$, we denote $h^+ = h_+$ and $h^- = h_-$. Otherwise, we denote $h^+ = h_-$ and $h^- = h_+$. Moreover, we set $\deg(0) = -1$.

For $a, b, c, d \in \mathbb{N}$, we define $ax^b > cx^d$ if $b > d$, or $b = d$ and $a > c$. For $h \in \mathbb{Z}[x]$, we use $\text{lt}(h)$ and $\text{lc}(h)$ to denote the leading term of h and the leading coefficient of h respectively.

Theorem 3.1. *For any $\mathbb{Z}[x]$ -lattice $L \subseteq \mathbb{Z}[x]^n$, $\langle \mathcal{I}_L \rangle_r$ is finitely generated as a radical well-mixed σ -ideal.*

Proof. Denote the set of all maps from $\{1, \dots, n\}$ to $\{+, -, 0\}$ by Λ and $\tau_0 \in \Lambda$ is the map such that $\tau_0(i) = 0$ for $1 \leq i \leq n$. Let $\Lambda_0 = \Lambda \setminus \{\tau_0\}$. For any $\tau \in \Lambda_0$, define

$$A_\tau := \{(h_1, \dots, h_n) \in L \mid \text{lc}(h_i) > 0 \text{ if } \tau(i) = +, \text{lc}(h_i) < 0 \text{ if } \tau(i) = -, \text{ and } \text{lc}(h_i) = 0 \text{ if } \tau(i) = 0, i = 1, \dots, n\},$$

and

$$\Sigma_\tau := \{(\deg(h_1^+), \text{lc}(h_1^+), \dots, \deg(h_n^+), \text{lc}(h_n^+), \deg(h_1^-), \dots, \deg(h_n^-)) \mid (h_1, \dots, h_n) \in A_\tau\}.$$

For any $\tau \in \Lambda_0$, let G_τ be the subset of A_τ such that

$$\{(\deg(h_1^+), \text{lc}(h_1^+), \dots, \deg(h_n^+), \text{lc}(h_n^+), \deg(g_1^-), \dots, \deg(g_n^-)) \mid \mathbf{g} = (g_1, \dots, g_n) \in G_\tau\}$$

is the set of minimal elements in Σ_τ under the product order. It follows that G_τ is a finite set. Let

$$F_\tau := \{\mathbb{Y}^{\mathbf{g}^+} - \mathbb{Y}^{\mathbf{g}^-} \mid \mathbf{g} \in G_\tau\}.$$

We claim that the finite set $\cup_{\tau \in \Lambda_0} F_\tau$ generates $\langle \mathcal{I}_L \rangle_r$ as a radical well-mixed σ -ideal.

Denote $\mathcal{I}_0 = \langle \cup_{\tau \in \Lambda_0} F_\tau \rangle_r$. We will prove the claim by showing that $\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-} \in \mathcal{I}_0$ for all $\mathbf{h} \in L$. Let us do induction on $(\text{lt}(h_1^+), \dots, \text{lt}(h_n^+))$ under the lexicographic order for $\mathbf{h} = (h_1, \dots, h_n) \in L$. For the simplicity, we will assume that $\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}$ has the form

$$y_1^{h_1^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} - y_1^{h_1^-} \cdots y_t^{h_t^-} y_{t+1}^{h_{t+1}^+} \cdots y_n^{h_n^+},$$

where $1 \leq t \leq n$. And without loss of generality, we furthermore assume $\text{lc}(h_i) \neq 0$ for $1 \leq i \leq n$.

The case for $\mathbf{h} = \mathbf{0}$ is trivial. Now for the inductive step. By definition, there exists $\tau \in \Lambda_0$ and $(g_1, \dots, g_n) \in G_\tau$ such that $(h_1, \dots, h_n) \in A_\tau$ and $\deg(g_i^+) \leq \deg(h_i^+), \text{lc}(g_i^+) \leq \text{lc}(h_i^+), \deg(g_i^-) \leq \deg(h_i^-), i = 1, \dots, n$. Choose $j \in \{1, \dots, n\}$ such that

$$\deg(h_j^+) - \deg(g_j^+) = \min_{1 \leq i \leq n} \{\deg(h_i^+) - \deg(g_i^+)\}.$$

Without loss of generality, we can assume $j = 1$. Let $s = \deg(h_1^+) - \deg(g_1^+) \geq 0$. Since $\text{lc}(h_1^+) \geq \text{lc}(g_1^+)$, there exists an $e \in \mathbb{N}[x]$ such that $\deg(e) < \deg(h_1^+)$ and $p := h_1^+ + e - x^s g_1^+ \in$

$\mathbb{N}[x]$ with $\text{lt}(p) < \text{lt}(h_1^+)$. Then

$$\begin{aligned}
& y_1^e y_2^{x^s g_2^+} \cdots y_t^{x^s g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (y_1^{h_1^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} - y_1^{h_1^-} \cdots y_t^{h_t^-} y_{t+1}^{h_{t+1}^+} \cdots y_n^{h_n^+}) \\
&= y_1^{p+x^s g_1^+} y_2^{h_2^+ + x^s g_2^+} \cdots y_t^{h_t^+ + x^s g_t^+} y_{t+1}^{h_{t+1}^- + x^s g_{t+1}^-} \cdots y_n^{h_n^- + x^s g_n^-} \\
&\quad - y_1^{h_1^- + e} y_2^{h_2^- + x^s g_2^+} \cdots y_t^{h_t^- + x^s g_t^+} y_{t+1}^{h_{t+1}^+ + x^s g_{t+1}^-} \cdots y_n^{h_n^+ + x^s g_n^-} \\
&= (y_1^{g_1^+} \cdots y_t^{g_t^+} y_{t+1}^{g_{t+1}^-} \cdots y_n^{g_n^-} - y_1^{g_1^-} \cdots y_t^{g_t^-} y_{t+1}^{g_{t+1}^+} \cdots y_n^{g_n^+}) x^s y_1^p y_2^{h_2^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} \\
&\quad + y_1^{p+x^s g_1^-} y_2^{h_2^+ + x^s g_2^-} \cdots y_t^{h_t^+ + x^s g_t^-} y_{t+1}^{h_{t+1}^- + x^s g_{t+1}^+} \cdots y_n^{h_n^- + x^s g_n^+} \\
&\quad - y_1^{h_1^- + e} y_2^{h_2^- + x^s g_2^+} \cdots y_t^{h_t^- + x^s g_t^+} y_{t+1}^{h_{t+1}^+ + x^s g_{t+1}^-} \cdots y_n^{h_n^+ + x^s g_n^-} \\
&= (y_1^{g_1^+} \cdots y_t^{g_t^+} y_{t+1}^{g_{t+1}^-} \cdots y_n^{g_n^-} - y_1^{g_1^-} \cdots y_t^{g_t^-} y_{t+1}^{g_{t+1}^+} \cdots y_n^{g_n^+}) x^s y_1^p y_2^{h_2^+} \cdots y_t^{h_t^+} y_{t+1}^{h_{t+1}^-} \cdots y_n^{h_n^-} \\
&\quad + y_1^{d_1} \cdots y_n^{d_n} (y_1^{w_1^+} \cdots y_n^{w_n^+} - y_1^{w_1^-} \cdots y_n^{w_n^-}).
\end{aligned}$$

Since $\text{lt}(p + x^s g_1^-) < \text{lt}(h_1^+)$, $\text{lt}(h_1^- + e) < \text{lt}(h_1^+)$, then $\text{lt}(w_1^+) < \text{lt}(h_1^+)$ and because of the choice of j , we have $s + \deg(g_i^+) \leq \deg(h_i^+)$ for $2 \leq i \leq n$, from which it follows $\text{lt}(w_i^+) \leq \text{lt}(h_i^+)$, $2 \leq i \leq n$. Therefore, $(\text{lt}(w_1^+), \dots, \text{lt}(w_n^+)) < (\text{lt}(h_1^+), \dots, \text{lt}(h_n^+))$. Thus by the induction hypothesis, $y_1^{w_1^+} \cdots y_n^{w_n^+} - y_1^{w_1^-} \cdots y_n^{w_n^-} \in \mathcal{I}_0$ and hence

$$y_1^e y_2^{x^s g_2^+} \cdots y_t^{x^s g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0.$$

So by the properties of radical well-mixed σ -ideals, we have

$$y_1^{x^s g_1^+} \cdots y_t^{x^s g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0,$$

and then

$$y_1^{x^s g_1^-} \cdots y_t^{x^s g_t^-} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0.$$

If $s > 0$, let $s' = \max\{0, s - \min_{1 \leq i \leq t} \{\deg(g_i^+) - \deg(g_i^-)\}\} < s$. Again by the properties of radical well-mixed σ -ideals, we have

$$y_1^{x^{s'} g_1^+} \cdots y_t^{x^{s'} g_t^+} y_{t+1}^{x^s g_{t+1}^-} \cdots y_n^{x^s g_n^-} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0,$$

and then

$$y_1^{x^{s'} g_1^-} \cdots y_t^{x^{s'} g_t^-} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0.$$

If $s' > 0$, repeat the above process, and we eventually obtain

$$y_1^{g_1^-} \cdots y_t^{g_t^-} y_{t+1}^{x^s g_{t+1}^+} \cdots y_n^{x^s g_n^+} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0.$$

Since $\deg(g_i^-) \leq \deg(h_i^-)$, $1 \leq i \leq t$ and $s + \deg(g_i^+) \leq \deg(h_i^+)$, $t+1 \leq i \leq n$, then by the properties of radical well-mixed σ -ideals, we have

$$(2) \quad \mathbb{Y}^{\mathbf{h}^-} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0.$$

Similarly, we also have

$$(3) \quad \mathbb{Y}^{\mathbf{h}^+} (\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-}) \in \mathcal{I}_0.$$

Combining (2) and (3), we obtain $(\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-})^2 \in \mathcal{I}_0$, and hence $\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-} \in \mathcal{I}_0$. So we complete the proof. \square

Corollary 3.2. *Let $L \subseteq \mathbb{Z}[x]^n$ be a $\mathbb{Z}[x]$ -lattice such that \mathcal{I}_L is well-mixed, then \mathcal{I}_L is finitely generated as a radical well-mixed σ -ideal.*

Proof. It is immediate from Theorem 3.1 since \mathcal{I}_L is already a radical well-mixed σ -ideal. \square

Example 3.3. *Let $L = \left(\begin{pmatrix} x-1 \\ 1-x \end{pmatrix}\right) \subseteq \mathbb{Z}[x]^2$ be a $\mathbb{Z}[x]$ -lattice. \mathcal{I}_L is a σ -prime σ -ideal. Then $\mathcal{I}_L = [y_1^{x^i} y_2 - y_1 y_2^{x^i} : i \in \mathbb{N}^*] = \langle y_1^x y_2 - y_1 y_2^x \rangle_r$.*

Example 3.4. *Let $L = \left(\begin{pmatrix} x^2+1-x \\ x-x^2-1 \end{pmatrix}\right) \subseteq \mathbb{Z}[x]^2$ be a $\mathbb{Z}[x]$ -lattice. \mathcal{I}_L is a σ -prime σ -ideal. Then $\mathcal{I}_L = \langle y_1^{x^2+1} y_2^x - y_1^x y_2^{x^2+1}, y_1^{x^3+1} - y_2^{x^3+1} \rangle_r$.*

Example 3.5. *Let $L = \left(\begin{pmatrix} x^2+1-x \\ x-1 \end{pmatrix}\right) \subseteq \mathbb{Z}[x]^2$ be a $\mathbb{Z}[x]$ -lattice. \mathcal{I}_L is a σ -prime σ -ideal. Then $\mathcal{I}_L = \langle y_1^{x^2+1} y_2^x - y_1^x y_2, y_1^{x^3+1} y_2^{x^2} - y_2 \rangle_r$.*

To show radical well-mixed σ -ideals generated by any binomials are finitely generated, we need the following lemma.

Lemma 3.6 ([7], Proposition 5.2). *Let F and G be subsets of any σ -ring R . Then*

$$\langle F \rangle_r \cap \langle G \rangle_r = \langle FG \rangle_r.$$

As a corollary, if I and J are two σ -ideals of R , then

$$\langle I \rangle_r \cap \langle J \rangle_r = \langle I \cap J \rangle_r = \langle IJ \rangle_r.$$

Proof. For the proof, please refer to [7]. \square

Lemma 3.7. *Suppose $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Then $\langle I \rangle_r : \mathfrak{m}$ is finitely generated as a radical well-mixed σ -ideal.*

Proof. Since $I : \mathfrak{m}$ is a normal binomial σ -ideal, there exists a $\mathbb{Z}[x]$ -lattice L such that $I : \mathfrak{m} = \mathcal{I}_L$. Note that $\langle I \rangle_r : \mathfrak{m} = \langle I : \mathfrak{m} \rangle_r : \mathfrak{m}$, so by Lemma 2.5, $\langle I \rangle_r : \mathfrak{m}$ is $[1]$ or $\mathcal{I}_{\text{sat}_M(L)}$. Since $\langle I \rangle_r$ is radical well-mixed, it is easy to show that $\langle I \rangle_r : \mathfrak{m}$ is radical well-mixed. So by Corollary 3.2, $\langle I \rangle_r : \mathfrak{m}$ is finitely generated as a radical well-mixed σ -ideal. \square

Lemma 3.8. *Suppose $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Then*

$$\langle I \rangle_r = \langle I \rangle_r : \mathfrak{m} \cap \langle I, y_{p_1}^{x^{a_1}} \rangle_r \cap \dots \cap \langle I, y_{p_l}^{x^{a_l}} \rangle_r$$

for some $\{p_1, \dots, p_l\} \subseteq \{1, \dots, n\}$ and some $(a_1, \dots, a_l) \in \mathbb{N}^l$.

Proof. By Lemma 3.7, $\langle I \rangle_r : \mathfrak{m}$ is finitely generated as a radical well-mixed σ -ideal. Therefore, there exist $f_1, \dots, f_s \in \langle I \rangle_r : \mathfrak{m}$ and $m_1, \dots, m_s \in \mathfrak{m}$ such that $\langle I \rangle_r : \mathfrak{m} = \langle f_1, \dots, f_s \rangle_r$

and $m_1 f_1, \dots, m_s f_s \in \langle I \rangle_r$. Then by Lemma 3.6,

$$\begin{aligned}
 \langle I \rangle_r &= \langle I, f_1 \rangle_r \cap \langle I, m_1 \rangle_r \\
 &= \langle I, f_1, f_2 \rangle_r \cap \langle I, f_1, m_2 \rangle_r \cap \langle I, m_1 \rangle_r \\
 &= \langle I, f_1, f_2 \rangle_r \cap \langle I, m_1 m_2 \rangle_r \\
 &= \dots \\
 &= \langle f_1, \dots, f_s \rangle_r \cap \langle I, m_1 \dots m_s \rangle_r \\
 &= \langle I \rangle_r : \mathfrak{m} \cap \langle I, y_{p_1}^{x^{a_1}} \rangle_r \cap \dots \cap \langle I, y_{p_l}^{x^{a_l}} \rangle_r,
 \end{aligned}$$

for some $\{p_1, \dots, p_l\} \subseteq \{1, \dots, n\}$ and some $(a_1, \dots, a_l) \in \mathbb{N}^l$. \square

Suppose $\{j_1, \dots, j_t\} \subseteq \{1, \dots, n\}$, $(a_1, \dots, a_t) \in \mathbb{N}^t$ and $I_0 \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Denote $T_{j_1 \dots j_t}^{a_1 \dots a_t} := \{y_1^{f_1} \dots y_n^{f_n} : f_1, \dots, f_n \in \mathbb{N}[x], \deg(f_{j_i}) < a_i, 1 \leq i \leq t\}$. We say I_0 is *saturated* with respect to $\{y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}\}$ if $I_0 = I_0 : T_{j_1 \dots j_t}^{a_1 \dots a_t}$, that is, for $g \in K\{y_1, \dots, y_n\}$ and $M \in T_{j_1 \dots j_t}^{a_1 \dots a_t}$, $Mg \in I_0$ implies $g \in I_0$. Let $I \subseteq K\{y_1, \dots, y_n\}$ be a pure binomial σ -ideal. The minimal σ -ideal containing I which is saturated with respect to $\{y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}\}$ is called the $T_{j_1 \dots j_t}^{a_1 \dots a_t}$ -*saturated closure* of I , denoted by $N_{j_1 \dots j_t}^{a_1 \dots a_t}(I)$. We will give a concrete description of the $T_{j_1 \dots j_t}^{a_1 \dots a_t}$ -saturated closure of a pure binomial σ -ideal I . Let $I^{[0]} = I$ and recursively define $I^{[i]} = [I^{[i-1]} : T_{j_1 \dots j_t}^{a_1 \dots a_t}]$ ($i = 1, 2, \dots$). The following lemma is easy to check by definition.

Lemma 3.9. *Let $I \subseteq K\{y_1, \dots, y_n\}$ be a pure binomial σ -ideal. Then*

$$(4) \quad N_{j_1 \dots j_t}^{a_1 \dots a_t}(I) = \bigcup_{i=0}^{\infty} I^{[i]}.$$

Let $I_0 \subseteq K\{y_1, \dots, y_n\}$ be a pure binomial σ -ideal. Then we say $I = \langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$ is *quasi-normal* if I_0 is saturated with respect to $\{y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}\}$ and for any binomial $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}} \in I_0$, if $\mathbb{Y}^{\mathbf{f}} \in [y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}]$, then $\mathbb{Y}^{\mathbf{g}} \in [y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}]$. Similarly to Theorem 3.1, we can prove the following useful lemma.

Lemma 3.10. *Let $\{j_1, \dots, j_t\} \subseteq \{1, \dots, n\}$, $(a_1, \dots, a_t) \in \mathbb{N}^t$ and $I_0 \subseteq K\{y_1, \dots, y_n\}$ a pure binomial σ -ideal. Assume that $I = \langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$ is quasi-normal. Then I is finitely generated as a radical well-mixed σ -ideal.*

Proof. Let $J = \{\mathbb{Y}^{\mathbf{h}^+} - \mathbb{Y}^{\mathbf{h}^-} \in I_0 \mid \mathbb{Y}^{\mathbf{h}^+}, \mathbb{Y}^{\mathbf{h}^-} \in T_{j_1 \dots j_t}^{a_1 \dots a_t}\}$. Similarly to Theorem 3.1, we can prove $\langle J \rangle_r$ is finitely generated as a radical well-mixed σ -ideal. Thus $I = \langle J, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$ is finitely generated as a radical well-mixed σ -ideal. \square

Lemma 3.11. *Suppose $\{j_1, \dots, j_t\} \subseteq \{1, \dots, n\}$, $(a_1, \dots, a_t) \in \mathbb{N}^t$ and $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Let $I_0 = N_{j_1 \dots j_t}^{a_1 \dots a_t}(I)$. Assume that $\langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$ is quasi-normal. Then*

$$\langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r = \langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \cap \bigcap_{1 \leq k \leq l} \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}} \rangle_r,$$

where either $p_k \notin \{j_1, \dots, j_t\}$, or $p_k = j_m$ and $b_k < a_m$ for $1 \leq k \leq l$.

Proof. Since $\langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$ is quasi-normal, by Lemma 3.10, it is finitely generated as a radical well-mixed σ -ideal. That is to say, there exist $f_1, \dots, f_s \in I_0$ such that

$$\langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r = \langle f_1, \dots, f_s, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r.$$

By (4), $I_0 = \cup_{i=0}^{\infty} I^{[i]}$, so there exists $i \in \mathbb{N}$ such that $f_1, \dots, f_s \in I^{[i]}$. By definition, there exist $g_{i1}, \dots, g_{il_i} \in I^{[i-1]} : T_{j_1 \dots j_t}^{a_1 \dots a_t}$ and $m_{i1}, \dots, m_{il_i} \in T_{j_1 \dots j_t}^{a_1 \dots a_t}$ such that $f_1, \dots, f_s \in [g_{i1}, \dots, g_{il_i}]$ and $m_{i1}g_{i1}, \dots, m_{il_i}g_{il_i} \in I^{[i-1]}$. Again there exist $g_{i-11}, \dots, g_{i-1l_{i-1}} \in I^{[i-2]} : T_{j_1 \dots j_t}^{a_1 \dots a_t}$ and $m_{i-11}, \dots, m_{i-1l_{i-1}} \in T_{j_1 \dots j_t}^{a_1 \dots a_t}$ such that $m_{i-11}g_{i-11}, \dots, m_{i-1l_{i-1}}g_{i-1l_{i-1}} \in [g_{i-11}, \dots, g_{i-1l_{i-1}}]$ and $m_{i-11}g_{i-11}, \dots, m_{i-1l_{i-1}}g_{i-1l_{i-1}} \in I^{[i-2]}$. Iterating this process, we eventually have there exist $g_{11}, \dots, g_{1l_1} \in I : T_{j_1 \dots j_t}^{a_1 \dots a_t}$ and $m_{11}, \dots, m_{1l_1} \in T_{j_1 \dots j_t}^{a_1 \dots a_t}$ such that $m_{21}g_{21}, \dots, m_{2l_2}g_{2l_2} \in [g_{11}, \dots, g_{1l_1}]$ and $m_{11}g_{11}, \dots, m_{1l_1}g_{1l_1} \in I$. So by Lemma 3.6, we obtain

$$\begin{aligned} \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r &= \langle I, g_{11}, \dots, g_{1l_1}, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \cap \langle I, m_{11} \cdots m_{1l_1}, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \\ &= \langle I, g_{21}, \dots, g_{2l_2}, g_{11}, \dots, g_{1l_1}, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \\ &\quad \cap \langle I, m_{21} \cdots m_{2l_2} m_{11} \cdots m_{1l_1}, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \\ &= \dots \\ &= \langle I, g_{i1}, \dots, g_{il_i}, \dots, g_{11}, \dots, g_{1l_1}, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \\ &\quad \cap \langle I, m_{i1} \cdots m_{il_i} \cdots m_{11} \cdots m_{1l_1}, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \\ &= \langle I_0, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \cap \bigcap_{1 \leq k \leq l} \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}} \rangle_r, \end{aligned}$$

where either $p_k \notin \{j_1, \dots, j_t\}$, or $p_k = j_m$ and $b_k < a_m$ for $1 \leq k \leq l$. \square

From the proof of Lemma 3.11, we obtain the following lemma which will be used later.

Lemma 3.12. Suppose $\{j_1, \dots, j_t\} \subseteq \{1, \dots, n\}$, $(a_1, \dots, a_t) \in \mathbb{N}^t$ and $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Let $h \in N_{j_1 \dots j_t}^{a_1 \dots a_t}(I)$. Then

$$\langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r = \langle I', y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r \cap \bigcap_{1 \leq k \leq l} \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}} \rangle_r,$$

where $I' \supseteq [I, h]$ and either $p_k \notin \{j_1, \dots, j_t\}$, or $p_k = j_m$ and $b_k < a_m$ for $1 \leq k \leq l$.

Lemma 3.13. Suppose $\{j_1, \dots, j_t\} \subseteq \{1, \dots, n\}$, $(a_1, \dots, a_t) \in \mathbb{N}^t$ and $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Assume that there exists a binomial $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}} \in I$ such that $\mathbb{Y}^{\mathbf{f}} \in [y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}]$ and $\mathbb{Y}^{\mathbf{g}} \notin [y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}]$. Then

$$\langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r = \bigcap_{1 \leq k \leq l} \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}} \rangle_r,$$

where either $p_k \notin \{j_1, \dots, j_t\}$, or $p_k = j_m$ and $b_k < a_m$ for $1 \leq k \leq l$.

Proof. Since there exists a binomial $\mathbb{Y}^{\mathbf{f}} - \mathbb{Y}^{\mathbf{g}} \in I$ such that $\mathbb{Y}^{\mathbf{f}} \in [y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}]$ and $\mathbb{Y}^{\mathbf{g}} \notin [y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}]$, then $\mathbb{Y}^{\mathbf{g}} \in \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$. Therefore, by the properties of radical well-mixed σ -ideals, there exist $\{p_1, \dots, p_l\} \subseteq \{1, \dots, n\}$ and $(b_1, \dots, b_l) \in \mathbb{N}^l$ satisfying either $p_k \notin \{j_1, \dots, j_t\}$, or $p_k = j_m$ and $b_k < a_m$, for $1 \leq k \leq l$ such that

$y_{p_1}^{x^{b_1}} \cdots y_{p_l}^{x^{b_l}} \in \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r$. Hence,

$$\langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}} \rangle_r = \bigcap_{1 \leq k \leq l} \langle I, y_{j_1}^{x^{a_1}}, \dots, y_{j_t}^{x^{a_t}}, y_{p_k}^{x^{b_k}} \rangle_r.$$

□

Lemma 3.14. *Let $i \in \{1, \dots, n\}$ and $a \in \mathbb{N}$. Suppose $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Then*

$$\langle I, y_i^{x^a} \rangle_r = \bigcap_{(j_1, \dots, j_t), (b_{j_1}, \dots, b_{j_t})} \langle I_{j_1, \dots, j_t}^{b_{j_1} \dots b_{j_t}}, y_{j_1}^{x^{b_{j_1}}}, \dots, y_{j_t}^{x^{b_{j_t}}} \rangle_r$$

is a finite intersection, $i \in \{j_1, \dots, j_t\}$ and for each member in the intersection, either $I_{j_1, \dots, j_t}^{b_{j_1} \dots b_{j_t}} \subseteq [y_{j_1}^{x^{b_{j_1}}}, \dots, y_{j_t}^{x^{b_{j_t}}}]$, or $\langle I_{j_1, \dots, j_t}^{b_{j_1} \dots b_{j_t}}, y_{j_1}^{x^{b_{j_1}}}, \dots, y_{j_t}^{x^{b_{j_t}}} \rangle_r$ is quasi-normal.

Proof. Using Lemma 3.13 repeatedly if there exists a binomial $\mathbb{Y}^f - \mathbb{Y}^g \in I$ such that $\mathbb{Y}^f \in [y_i^{x^a}]$ and $\mathbb{Y}^g \notin [y_i^{x^a}]$, assume that we obtain a decomposition as follows:

$$(5) \quad \langle I, y_i^{x^a} \rangle_r = \bigcap_{(j_1 \dots j_t), (c_{j_1} \dots c_{j_t})} \langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r.$$

For each $I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}$, if $I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}} \subseteq [y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}]$, then we have nothing to do. Otherwise, let $I_0 = N_{j_1 \dots j_t}^{c_{j_1} \dots c_{j_t}}(I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}})$. If there exists a binomial $\mathbb{Y}^f - \mathbb{Y}^g \in I_0$ such that $\mathbb{Y}^f \in [y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}]$ and $\mathbb{Y}^g \notin [y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}]$. Then by Lemma 3.12,

$$\langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r = \langle I', y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r \cap \bigcap_{1 \leq k \leq l} \langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}, y_{p_k}^{x^{d_k}} \rangle_r,$$

where $I' \supseteq [I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, \mathbb{Y}^f - \mathbb{Y}^g]$ and either $p_k \notin \{j_1, \dots, j_t\}$, or $p_k = j_m$ and $d_k < c_{j_m}$ for $1 \leq k \leq l$. Therefore, by Lemma 3.13, we have

$$\langle I', y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r = \bigcap_{1 \leq k \leq l'} \langle I', y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}, y_{s_k}^{x^{e_k}} \rangle_r,$$

where either $s_k \notin \{j_1, \dots, j_t\}$, or $s_k = j_m$ and $e_k < c_{j_m}$ for $1 \leq k \leq l'$. Thus

$$(6) \quad \langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r = \bigcap_{1 \leq k \leq l'} \langle I', y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}, y_{s_k}^{x^{e_k}} \rangle_r \cap \bigcap_{1 \leq k \leq l} \langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}, y_{p_k}^{x^{d_k}} \rangle_r.$$

For each member in the intersection (6), repeat the above process. Since at each step, either the number of elements of $\{y_{j_1}, \dots, y_{j_t}\}$ strictly increase, or the vector $(c_{j_1}, \dots, c_{j_t})$ strictly decrease, then in finite steps we must obtain either $I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}} \subseteq [y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}]$, or for any binomial $\mathbb{Y}^f - \mathbb{Y}^g \in I_0$, if $\mathbb{Y}^f \in [y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}]$, then $\mathbb{Y}^g \in [y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}]$. In the latter case, by Lemma 3.11,

$$\langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r = \langle I_0, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}} \rangle_r \cap \bigcap_{1 \leq k \leq l''} \langle I_{j_1, \dots, j_t}^{c_{j_1} \dots c_{j_t}}, y_{j_1}^{x^{c_{j_1}}}, \dots, y_{j_t}^{x^{c_{j_t}}}, y_{t_k}^{x^{b_k}} \rangle_r,$$

where either $t_k \notin \{j_1, \dots, j_t\}$, or $t_k = j_m$ and $b_k < c_{j_m}$ for $1 \leq k \leq l''$.

Apply the same procedure to the rest of the members in the intersection, and in finite steps we obtain the desired decomposition. \square

Now we can prove the main theorem of this paper.

Theorem 3.15. *Suppose $I \subseteq K\{y_1, \dots, y_n\}$ is a pure binomial σ -ideal. Then $\langle I \rangle_r$ is finitely generated as a radical well-mixed σ -ideal.*

Proof. By Lemma 3.8, we have

$$(7) \quad \langle I \rangle_r = \langle I \rangle_r : \mathfrak{m} \cap \langle I, y_{p_1}^{x^{a_1}} \rangle_r \cap \dots \cap \langle I, y_{p_l}^{x^{a_l}} \rangle_r$$

for some $\{p_1, \dots, p_l\} \subseteq \{1, \dots, n\}$ and some $\{a_1, \dots, a_l\} \in \mathbb{N}^l$. By Lemma 3.14,

$$(8) \quad \langle I, y_{p_k}^{x^{a_k}} \rangle_r = \bigcap_{(j_1 \dots j_t), (b_{j_1} \dots b_{j_t})} \langle I_{j_1 \dots j_t}^{b_{j_1} \dots b_{j_t}}, y_{j_1}^{x^{b_{j_1}}}, \dots, y_{j_t}^{x^{b_{j_t}}} \rangle_r.$$

Since in (8), either $I_{j_1 \dots j_t}^{b_{j_1} \dots b_{j_t}} \subseteq [y_{j_1}^{x^{b_{j_1}}}, \dots, y_{j_t}^{x^{b_{j_t}}}]$, or $\langle I_{j_1 \dots j_t}^{b_{j_1} \dots b_{j_t}}, y_{j_1}^{x^{b_{j_1}}}, \dots, y_{j_t}^{x^{b_{j_t}}} \rangle_r$ is quasi-normal, then by Lemma 3.10, each member in the intersection (8) is finitely generated as a radical well-mixed σ -ideal. And since (8) is a finite intersection, by Lemma 3.6, $\langle I, y_{p_k}^{x^{a_k}} \rangle_r$ is finitely generated as a radical well-mixed σ -ideal for $1 \leq k \leq l$. Moreover, by Lemma 3.7, $\langle I \rangle_r : \mathfrak{m}$ is finitely generated as a radical well-mixed σ -ideal. Putting all above together, by (7) and Lemma 3.6, $\langle I \rangle_r$ is finitely generated as a radical well-mixed σ -ideal. \square

Corollary 3.16. *Any strictly ascending chain of radical well-mixed σ -ideals generated by pure binomials in a σ -polynomial ring is finite.*

Proof. Assume that $I_1 \subseteq I_2 \subseteq \dots \subseteq I_k \dots$ is an ascending chain of radical well-mixed σ -ideals generated by pure binomials in a σ -polynomial ring. Then $\bigcup_{i=1}^{\infty} I_i$ is also a radical well-mixed σ -ideal generated by pure binomials. By Theorem 3.15, $\bigcup_{i=1}^{\infty} I_i$ is finitely generated as a radical well-mixed σ -ideal, say $\{a_1, \dots, a_m\}$. Then there exists $k \in \mathbb{N}$ large enough such that $\{a_1, \dots, a_m\} \subset I_k$. It follows $I_k = I_{k+1} = \dots = \bigcup_{i=1}^{\infty} I_i$. \square

Remark 3.17. *By Corollary 3.16, Conjecture 1.1 is valid for radical well-mixed σ -ideals generated by pure binomials in a σ -polynomial ring.*

Remark 3.18. *Theorem 3.15 and Corollary 3.16 actually hold for radical well-mixed σ -ideals generated by any binomials (not necessarily pure binomials).*

In [6], A. Levin gave an example to show that a strictly ascending chain of well-mixed σ -ideals in a σ -polynomial ring may be infinite. Here we give a simpler example.

Example 3.19. *Let $I = \langle y_1^x y_2 - y_1 y_2^x \rangle$ and $I_0 = [y_1^x y_2 - y_1 y_2^x, y_1^{x^j} (y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l}, y_2^{x^j} (y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l} : i, j, l \in \mathbb{N}, i \geq 2, j \geq i - 1]$. We claim that $I = I_0$. It is easy to check that $I_0 \subseteq I$. So we only need to show that I_0 is already a well-mixed σ -ideal. Following Example 3.3, let $\mathcal{I}_L = \langle y_1^x y_2 - y_1 y_2^x \rangle_r$. Suppose $ab \in I_0 \subseteq \mathcal{I}_L$. Since $\mathcal{I}_L = [y_1^{x^i} y_2 - y_1 y_2^{x^i} : i \in \mathbb{N}^*]$ is a σ -prime σ -ideal, then $a \in \mathcal{I}_L$ or $b \in \mathcal{I}_L$. In each case, we can easily deduce that $ab^x \in I_0$. Therefore, I_0 is well-mixed and $I = I_0$. Thus $y_1^{x^2} y_2 - y_1 y_2^{x^2} \notin I$. In fact, in a similar way we can show that $\langle y_1^x y_2 - y_1 y_2^x, \dots, y_1^{x^k} y_2 - y_1 y_2^{x^k} \rangle = [y_1^x y_2 - y_1 y_2^x, \dots, y_1^{x^k} y_2 - y_1 y_2^{x^k}, y_1^{x^j} (y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l}, y_2^{x^j} (y_1^{x^i} y_2 - y_1 y_2^{x^i})^{x^l} : i, j, l \in \mathbb{N}, i \geq k + 1, j \geq i - k]$ and $y_1^{x^{k+1}} y_2 - y_1 y_2^{x^{k+1}} \notin$*

$\langle y_1^x y_2 - y_1 y_2^x, \dots, y_1^{x^k} y_2 - y_1 y_2^{x^k} \rangle$ for $k \geq 2$. So we obtain a strictly infinite ascending chain of well-mixed σ -ideals:

$$\langle y_1^x y_2 - y_1 y_2^x \rangle \subsetneq \langle y_1^x y_2 - y_1 y_2^x, y_1^{x^2} y_2 - y_1 y_2^{x^2} \rangle \subsetneq \dots \subsetneq \langle y_1^x y_2 - y_1 y_2^x, \dots, y_1^{x^k} y_2 - y_1 y_2^{x^k} \rangle \subsetneq \dots$$

As a consequence, \mathcal{I}_L is not finitely generated as a well-mixed σ -ideal.

In [2], it is shown that the radical closure, the reflexive closure, and the perfect closure of a binomial σ -ideal are still a binomial σ -ideal. However, the well-mixed closure of a binomial σ -ideal may not be a binomial σ -ideal. More precisely, it relies on the action of the difference operator. We will give an example to illustrate this.

Example 3.20. Let $K = \mathbb{C}$ and $R = \mathbb{C}\{y_1, y_2, y_3, y_4\}$. Let us consider the σ -ideal $I = \langle y_1^2(y_3 - y_4), y_2^2(y_3 - y_4) \rangle$ of R . Since $(y_1^2 - y_2^2)(y_3 - y_4) = (y_1 + y_2)(y_1 - y_2)(y_3 - y_4) \in I$, we have $(y_1 + y_2)(y_1 - y_2)^x(y_3 - y_4) = (y_1^{x+1} + y_1^x y_2 - y_1 y_2^x - y_2^{x+1})(y_3 - y_4) \in I$. Note that $y_1^{x+1}(y_3 - y_4), y_2^{x+1}(y_3 - y_4) \in I$, hence $(y_1^x y_2 - y_1 y_2^x)(y_3 - y_4) \in I$. If the difference operator on \mathbb{C} is the identity map, similarly to Example 4.1 in [7], we can show that $y_1^x y_2(y_3 - y_4), y_1 y_2^x(y_3 - y_4) \notin I$. As a consequence, I is not a binomial σ -ideal.

On the other hand, if the difference operator on \mathbb{C} is the conjugation map (that is $\sigma(i) = -i$), the situation is totally changed. Since $(y_1^2 + y_2^2)(y_3 - y_4) = (y_1 + iy_2)(y_1 - iy_2)(y_3 - y_4) \in I$, $(y_1 + iy_2)(y_1 - iy_2)^x(y_3 - y_4) = (y_1^{x+1} + iy_1^x y_2 + iy_1 y_2^x - y_2^{x+1})(y_3 - y_4) \in I$ and hence $(y_1^x y_2 + y_1 y_2^x)(y_3 - y_4) \in I$. Since we also have $(y_1^x y_2 - y_1 y_2^x)(y_3 - y_4) \in I$, then $y_1^x y_2(y_3 - y_4), y_1 y_2^x(y_3 - y_4) \in I$. Actually $I = [y_1^u(y_3 - y_4)^a, y_1^{w_1} y_2^{w_2}(y_3 - y_4)^a, y_2^v(y_3 - y_4)^a : u, v, w_1, w_2, a \in \mathbb{N}[x], 2 \preceq u, 2 \preceq v, x+1 \preceq w_1 + w_2]$ (\preceq is defined in [7]). In this case, $I = \langle y_1^2(y_3 - y_4), y_2^2(y_3 - y_4) \rangle$ is indeed a binomial σ -ideal.

Problem 3.21. We conjecture that the radical well-mixed closure of a binomial σ -ideal is still a binomial σ -ideal. However, we cannot prove it now.

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KLMM, ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE, THE CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

E-mail address: wangjie212@mails.ucas.ac.cn